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EQUATIONS OF THE SHALLOW WATER MODEL ON A ROTATING ATTRACTING SPHERE. 1. DERIVATION AND GENERAL PROPERTIES

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A shallow water model on a rotating attracting sphere is proposed to describe large-scale motions of the gas in planetary atmospheres and of the liquid in the world ocean. The equations of the model coincide with the equations of gas-dynamic of a polytropic gas in the case of spherical gas motions on the surface of a rotating sphere. The range of applicability of the model is discussed, and the conservation of potential vorticity along the trajectories is proved. The equations of stationary shallow water motions are presented in the form of Bernoulli and potential vorticity integrals, which relate the liquid depth to the stream function. The simplest stationary solutions that describe the equilibrium state differing from the spherically symmetric state and the zonal flows along the parallels are found. It is demonstrated that the stationary equations of the model admit the infinitely dimensional Lie group of equivalence.

Key words: *shallow water, motions on a sphere, Lie groups, potential vorticity, stationary solutions.*

Introduction. Description of hydrodynamic phenomena that occur in the atmosphere and in the ocean is a non-trivial problem. Important factors that affect the dynamics of the liquid or gaseous shell of a planet are gravitation and rotation. It is interaction of these two forces that retains the medium as a whole in an equilibrium state, with motions of different scales in the atmosphere.

For brevity, in what follows, we understand the atmosphere hydrodynamics as the motion of a continuous medium (liquid or gas) located on the surface of a rotating sphere in the field of a gravity force directed toward the sphere center with a constant acceleration. The model being developed is equally applicable to gas motions in planetary atmospheres and to large-scale oceanic flows.

Hydrodynamic phenomena in the atmosphere are characterized by a large variety of scales. On one hand, these are large-scale (planetary) phenomena (circulation cells), in particular, global vortices, such as cyclones and anticyclones, and oceanic flows; on the other hand, these are small-scale motions, which essentially depend on the surface relief. As the phenomenon under study depends on many factors, the universal model of atmosphere hydrodynamics is extremely complicated and difficult for investigations. It seems natural to identify the characteristic scales of motion to be studied and to use an appropriate approximate hydrodynamic model.

The specific feature of atmosphere hydrodynamics problems is the compactness of the manifold on which the corresponding mathematical model is determined. Available results on the behavior of the vector fields on a sphere, on one hand, and an empirical idea that all areas of the planet cannot have an identical weather, on the other hand, allow us to conclude that solutions that describe the motion of the atmosphere as a whole have singularities, such as sources and sinks, discontinuities and fronts separating air masses with different characteristics of motion.

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The basic model commonly used to describe large-scale phenomena in the atmosphere is the Euler equations for an ideal incompressible fluid. The assumptions of fluid incompressibility and a minor effect of viscosity are valid for moderate, obviously subsonic velocities of the medium and large-scale motions. The shallow water model derived from the problem with a free boundary for the Euler equations under the assumption that the vertical scale of motion is small, as compared with the horizontal scale, is used in atmosphere and ocean hydrodynamics. Further simplifications are usually applied to the shallow water model, and something similar to a model of the f- or β-plane is considered. Some part of the sphere surface is replaced by a plane tangential to it, which substantially simplifies the model. This approximation, however, cannot be used to describe the motion on a sphere, which is extended over the latitude. In this case, full equations of the shallow water theory on a sphere should be considered in this case.

Various hydrodynamic phenomena in the atmosphere and in the ocean have been studied in numerous papers (see, e.g., [1–8]). It should be noted that the first papers dealing with hydrodynamics on a sphere were those written by Gromeka [9] and Zermelo [10].

Various versions of the shallow water equations are used in atmosphere and ocean hydrodynamics, but none of the papers describes systematic derivation of the model on a sphere as a whole. The shallow water theory on a sphere without rotation was justified in the paper of Ovsyannikov [11], where the model equations were derived by expanding the solution of the exact Cauchy–Poisson problem of waves in an ideal incompressible fluid on the sphere surface with respect to a small parameter. For physical applications and for evaluating the quality of approximation, it seems reasonable to derive shallow water equations directly from the Euler equations.

The quantity $\varepsilon = H_0/a_0$ plays the role of the small parameter in shallow water equations [H_0 is the characteristic depth of the layer occupied by the continuous medium and a_0 is the sphere (planet) radius]. Shallow water equations are derived under the assumption that the parameter ε is small, the field of velocities is independent of the vertical (radial) coordinate, and there is no vertical transfer (the radial component of velocity equals zero).

Thus, the model describes large-scale planetary motions that occur during time periods of the order of several weeks. For such time scales, planet rotation exerts a significant effect on medium motion. In this case, the velocity can be interpreted as its mean value averaged over the depth, and the temperature effects are assumed to be manifested via the motion hydrostaticity only.

In the present paper, we derive equations of the shallow water model on a rotating attracting sphere and study some of its general properties: conservation of potential vorticity, the form of the solution of equations of stationary motions in terms of the stream function, the presence of an infinitely dimensional Lie group of equivalence of stationary equations, as well as stationary zonal flows.

1. Equations of Motion in a Rotating Coordinate System. We formulate the system of equations in a non-inertial coordinate system fitted to the rotating planet. Let us briefly describe derivation of the equations of motion of a continuous medium in a coordinate system rotating with a constant angular velocity $\boldsymbol{\omega}$. This seems to be reasonable, because some terms are unduly omitted in deriving equations of motion in some books on atmosphere hydrodynamics.

Let $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ be the orths of a certain inertial coordinate system, and let $\boldsymbol{i}'$, $\boldsymbol{j}'$, and $\boldsymbol{k}'$ be the orths fitted to a rotating planet. Then, we obtain the following expansions for an arbitrary vector $\boldsymbol{a}$:

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} = a_1'\boldsymbol{i}' + a_2'\boldsymbol{j}' + a_3'\boldsymbol{k}'. \tag{1.1}$$

As we have

$$\frac{d\boldsymbol{i}'}{dt} = \boldsymbol{\omega} \times \boldsymbol{i}', \qquad \frac{d\boldsymbol{j}'}{dt} = \boldsymbol{\omega} \times \boldsymbol{j}', \qquad \frac{d\boldsymbol{k}'}{dt} = \boldsymbol{\omega} \times \boldsymbol{k}' \tag{1.2}$$

in the rotating coordinate system, Eq. (1.1) yields

$$\frac{d\boldsymbol{a}}{dt} = \frac{d'\boldsymbol{a}}{dt} + \boldsymbol{\omega} \times \boldsymbol{a}, \tag{1.3}$$

where d'/dt is the total derivative in the non-inertial coordinate system.

Let $\boldsymbol{a} = \boldsymbol{x}$ be the radius vector:

$$\boldsymbol{x} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = x'\boldsymbol{i}' + y'\boldsymbol{j}' + z'\boldsymbol{k}'.$$

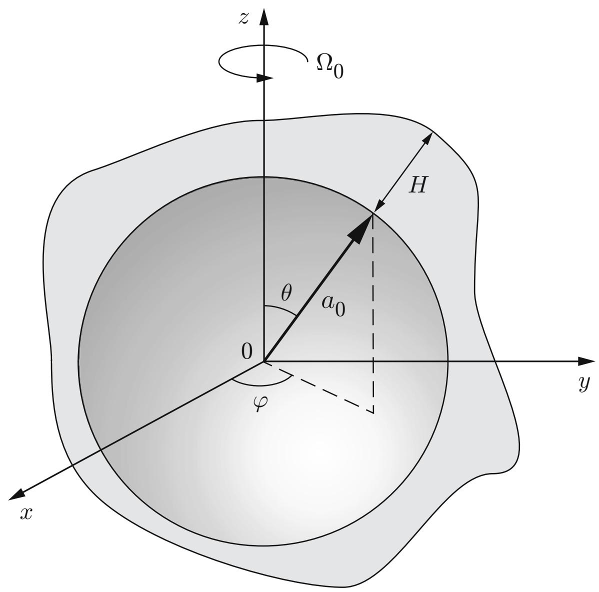

Fig. 1. Layout of the problem and coordinate system.

Then, $\dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{u}$ and $\dfrac{d'\boldsymbol{x}}{dt} = \boldsymbol{u}'$ are the velocities in the inertial and non-inertial coordinate systems, respectively. For $\boldsymbol{a} = \boldsymbol{x}$ and $\boldsymbol{u} = d\boldsymbol{x}/dt$, Eqs. (1.1) and (1.3) yield

$$\boldsymbol{u} = \boldsymbol{u}' + \boldsymbol{\omega} \times \boldsymbol{x}. \tag{1.4}$$

Let $\boldsymbol{a} = \boldsymbol{u}$ in Eq. (1.3). Then we have

$$\frac{d\boldsymbol{u}}{dt} = \frac{d'\boldsymbol{u}}{dt} + \boldsymbol{\omega} \times \boldsymbol{u}. \tag{1.5}$$

Substituting Eq. (1.4) into Eq. (1.5), we obtain

$$\frac{d\boldsymbol{u}}{dt} = \frac{d'\boldsymbol{u}'}{dt} + 2\boldsymbol{\omega} \times \boldsymbol{u}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{x}).$$

Thus, the equations of motion in the rotating planet-fitted coordinate system are written in the form

$$\frac{d'\boldsymbol{u}'}{dt} + 2\boldsymbol{\omega} \times \boldsymbol{u}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{x}) = \boldsymbol{g} - \rho^{-1}\nabla p, \tag{1.6}$$

where $\boldsymbol{g}$ is the free-fall acceleration. The term $2\boldsymbol{\omega} \times \boldsymbol{u}'$ is called the Coriolis force. Sometimes the centrifugal force $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{x})$ is united with the gravity force by introducing the acceleration $\boldsymbol{g}_1 = \boldsymbol{g} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{x})$; this procedure is justified by the fact that both these forces are functions of the point location only.

In some papers, the term corresponding to the centrifugal force in Eq. (1.6) is immediately omitted. This action is motivated by the smallness of this term, as compared with the Coriolis force, because it depends on the angular velocity squared. Generally speaking, this action is illegitimate before dimensionless parameters are introduced in Eqs. (1.6) and the magnitudes of terms in equations with dimensionless variables are estimated.

2. Formulation of the Exact Problem. By an example of a sphere of radius a_0, let us consider the motions in a layer of an ideal incompressible fluid $a_0 < r < a_0 + H(t, \theta, \varphi)$. Here $r = \sqrt{x^2 + y^2 + z^2}$; the latitude $0 < \theta < \pi$ and longitude $0 \leqslant \varphi < 2\pi$ are the spherical coordinates. The fluid is affected by the gravity force directed to the sphere center; g is a constant acceleration. The sphere rotates with a constant angular velocity Ω_0. Let us use U, V, and W to denote the radial, latitudinal (positive in the south direction), and longitudinal (positive in the east direction) components of the velocity vector $\boldsymbol{u}$; $R = \text{const}$ is the density, and P is the pressure in the fluid. The coordinate system is chosen in such a way that the axis of revolution coincides with the z axis passing through the north pole N ($\theta = 0$) and south pole S ($\theta = \pi$) (Fig. 1).

The Euler equations are valid inside the layer. Let us write these equations in a non-inertial coordinate system fitted to the rotating planet. According to Eq. (1.6), we have

$$
\begin{gathered}
DU = r^{-1}(V^2 + W^2) - R^{-1}P_r + 2\Omega_0 W \sin\theta + r\Omega_0^2 \sin^2\theta - g, \\
DV = r^{-1}(W^2 \cot\,\theta - UV) - (rR)^{-1}P_\theta + 2\Omega_0 W \cos\theta + r\Omega_0^2 \sin\theta\cos\theta, \\
DW = -r^{-1}W(V \cot\,\theta + U) - (rR\sin\theta)^{-1}P_\varphi + 2\Omega_0(U\sin\theta + V\cos\theta), \\
r^{-2}(r^2U)_r + (r\sin\theta)^{-1}(W_\varphi + (V\sin\theta)_\theta) = 0,
\end{gathered}
\tag{2.1}
$$

where $D = \partial_t + U\,\partial_r + r^{-1}V\,\partial_\theta + (r\sin\theta)^{-1}W\,\partial_\varphi$ is the total derivative. Let us introduce a variable $z = r - a_0$ $(z > 0)$. Then, we obtain

$$
r^{-1} = (z + a_0)^{-1} = a_0^{-1}\Big(1 + \frac{z}{a_0}\Big)^{-1} = \frac{1}{a_0}\Big(1 - \frac{z}{a_0} + \dots\Big). \tag{2.2}
$$

Let us assume that the quantity z/a_0 is small and omit all terms in the right side of Eq. (2.2) except for the first one, assuming that

$$
r^{-1} \approx a_0{}^{-1}, \qquad \partial_r = \partial_z. \tag{2.3}
$$

Taking into account assumptions (2.3), we can write system (2.1) in the form

$$
\begin{gathered}
D'U = a_0^{-1}(V^2 + W^2) - R^{-1}P_z + 2\Omega_0 W \sin\theta + a_0\Omega_0^2 \sin^2\theta - g, \\
D'V = a_0^{-1}(W^2 \cot\,\theta - UV) - (a_0R)^{-1}P_\theta + 2\Omega_0 W \cos\theta + a_0\Omega_0^2 \sin\theta\cos\theta, \\
D'W = -a_0^{-1}W(V \cot\,\theta + U) - (a_0R\sin\theta)^{-1}P_\varphi + 2\Omega_0(U\sin\theta + V\cos\theta), \\
U_z + a_0^{-1}U + (a_0\sin\theta)^{-1}(W_\varphi + (V\sin\theta)_\theta) = 0,
\end{gathered}
\tag{2.4}
$$

where $D' = \partial_t + U\,\partial_z + a_0^{-1}(V\,\partial_\theta + (\sin\theta)^{-1}W\,\partial_\varphi)$. Equations (2.4) are satisfied in the layer $0 < z < H(t, \theta, \varphi)$. On the bottom $(z = 0)$, we impose the condition

$$
U\Big|_{z=0} = 0. \tag{2.5}
$$

On the free boundary $z = H(t, \theta, \varphi)$, we set the dynamic condition $P = 0$ and the kinematic condition

$$
H_t + a_0^{-1}(VH_\theta + (\sin\theta)^{-1}WH_\varphi) = U. \tag{2.6}
$$

Problem (2.4)–(2.6) is considered as the initial problem, and it is from this problem that the shallow water equations on a rotating sphere will be derived.

3. Transition to Dimensionless Variables. Let us introduce dimensionless variables τ, z_1, u, v, w, ρ, p, and h related to the initial variables as follows:

$$
\begin{gathered}
t = T_0\tau, \quad z = H_0 z_1, \quad U = U_0 u, \quad V = V_0 v, \quad W = V_0 w, \\
R = R_0\rho, \quad P = P_0 p, \quad H = H_0 h.
\end{gathered}
\tag{3.1}
$$

The constants T_0, H_0, U_0, V_0, R_0, and P_0 in Eqs. (3.1) are the characteristic quantities that describe the time, the vertical scale, the radial and tangential components of velocity, the density, and the pressure. The latitudinal and longitudinal scales of velocity are assumed to be identical and equal to V_0. Passing to variables (3.1) in Eq. (2.4), we obtain the system

$$
\begin{gathered}
\bar{D}u = \frac{V_0^2 T_0}{a_0 U_0}\,(v^2 + w^2) - \frac{T_0P_0}{U_0R_0H_0}\,\rho^{-1}p_{z_1} + 2\Omega_0\,\frac{T_0V_0}{U_0}\,w\sin\theta - \frac{gT_0}{U_0} + \frac{a_0T_0\Omega_0^2}{U_0}\,\sin^2\theta, \\
\bar{D}v = \frac{V_0T_0}{a_0}\,w^2\cot\,\theta - \frac{U_0T_0}{a_0}\,uv - \frac{T_0P_0}{V_0R_0a_0}\,\rho^{-1}p_\theta + 2T_0\Omega_0 w\cos\theta + \frac{a_0T_0\Omega_0^2}{V_0}\,\sin\theta\cos\theta, \\
\bar{D}w = -\frac{V_0T_0}{a_0}\,vw\cot\,\theta - \frac{U_0T_0}{a_0}\,uw - \frac{T_0P_0}{V_0R_0a_0}\,(\rho\sin\theta)^{-1}p_\varphi - 2T_0\Omega_0 v\cos\theta - \frac{2U_0T_0}{V_0}\,\Omega_0 u\sin\theta, \\
\frac{U_0}{H_0}u_{z_1} + \frac{2U_0}{a_0}\,u + \frac{V_0}{a_0}\,(\sin\theta)^{-1}(w_\varphi + (v\sin\theta)_\theta) = 0,
\end{gathered}
\tag{3.2}
$$

where $\bar{D} = \partial_\tau + u\,\partial_{z_1} + v\,\partial_\theta + (\sin\theta)^{-1} w\,\partial_\varphi$. After normalization, the kinematic condition (2.6) acquires the form

$$h_\tau + \frac{T_0 V_0}{a_0}\,(v h_\theta + (\sin\theta)^{-1} w h_\varphi) = \frac{T_0 U_0}{H_0}\,u. \tag{3.3}$$

In the case of a rotating sphere, there is a natural time scale $T_0 = 2\pi/\Omega_0$. The characteristic lengths H_0 and a_0 and velocities U_0 and V_0 are related as

$$T_0 = \frac{H_0}{U_0} = \frac{a_0}{V_0}. \tag{3.4}$$

Relations (3.4) correspond to identical time scales of motions in the radial direction with the characteristic velocity U_0 at distances H_0 and in the tangential direction (along the sphere surface) with a velocity V_0 at distances of the order of a_0. Let us introduce the parameter of shallow water on a sphere

$$\varepsilon = H_0/a_0.$$

We denote

$$D_0' = \partial_\tau + u\,\partial_{z_1} + v\,\partial_\theta + (\sin\theta)^{-1} w\,\partial_\varphi. \tag{3.5}$$

Then, taking into account assumption (3.4), we write Eqs. (3.2) in the form

$$\begin{aligned}
\varepsilon^2 D_0' u &= \varepsilon\Big(v^2 + w^2 + \frac{2\Omega_0 a_0}{V_0}\,w\sin\theta + \Big(\frac{a_0\Omega_0}{V_0}\Big)^2 \sin^2\theta\Big) - \frac{P_0}{R_0 V_0^2}\Big(\rho^{-1} p_{z_1} + g\,\frac{R_0 H_0}{P_0}\Big),\\
D_0' v &= w^2 \cot\,\theta - \varepsilon u v + 2\,\frac{a_0\Omega_0}{V_0}\,w\cos\theta + \Big(\frac{a_0\Omega_0}{V_0}\Big)^2 \sin\theta\cos\theta - \frac{P_0}{R_0 V_0^2}\,\frac{p_\theta}{\rho},\\
D_0' w &= -v w \cot\,\theta - 2\,\frac{a_0\Omega_0}{V_0}\,v\cos\theta - \frac{P_0}{R_0 V_0^2}\,\frac{p_\varphi}{\sin\theta} - \varepsilon\Big(w + \frac{2a_0\Omega_0}{V_0}\,\sin\theta\Big)u,\\
& u_{z_1} + (\sin\theta)^{-1}(w_\varphi + (v\sin\theta)_\theta) + 2\varepsilon u = 0.
\end{aligned} \tag{3.6}$$

Let us assume that the parameter ε is small and the dimensionless complexes $a_0\Omega_0/V_0$ and $P_0/(R_0 V_0^2)$ in Eqs. (3.6) are finite. Then, omitting all terms containing ε and ε^2 in Eqs. (3.6), we use the first equation of system (3.6) to derive the condition of motion hydrostaticity

$$\rho^{-1} p_{z_1} + R_0 H_0 g/P_0 = 0. \tag{3.7}$$

Integrating Eq. (3.7) with respect to z_1 and taking into account the dynamic condition $p = 0$ on the free boundary, we obtain the relation for pressure:

$$p = -R_0 H_0 g (z_1 - h)\rho/P_0. \tag{3.8}$$

Equation (3.8) yields the formulas for the pressure gradient on the sphere

$$\rho^{-1} p_\theta = R_0 g H_0 h_\theta/P_0, \qquad \rho^{-1} p_\varphi = R_0 g H_0 h_\varphi/P_0. \tag{3.9}$$

The kinematic condition (3.3) acquires the form

$$h_\tau + v h_\theta + (\sin\theta)^{-1} w h_\varphi = u. \tag{3.10}$$

The last equation of system (3.6), which is the continuity equation, can be integrated with respect to z_1 under the assumption that the velocity components u and v are independent of the variable z_1:

$$u = -z_1 (\sin\theta)^{-1}(w_\varphi + (v\sin\theta)_\theta). \tag{3.11}$$

Under this assumption, the gas flow becomes shearless with respect to the vertical coordinate; therefore, the value of the velocity coordinate u can be taken as its value on the free boundary $z_1 = h$. Assuming that $z_1 = h$ in Eq. (3.11) and substituting Eq. (3.11) into Eq. (3.10), we obtain the equation

$$h_\tau + v h_\theta + (\sin\theta)^{-1} w h_\varphi + (\sin\theta)^{-1} h (w_\varphi + (v\sin\theta)_\theta) = 0. \tag{3.12}$$

The second and third equations of system (3.6) [after substitution of Eqs. (3.9) into Eqs. (3.6) and omitting terms containing ε] and Eq. (3.12) form a closed system of three equations for three functions h, v, and w of the independent variables τ, θ, and φ. In this case, the term $u\,\partial_{z_1}$ should be omitted in the total derivative (3.5).

Let us introduce the Rossby number (R_0) and Froude number (F)

$$R_0 = V_0/(2\Omega_0 a_0), \qquad F = V_0/\sqrt{gH_0},$$

which are the characteristic dimensionless parameters of the problem. For convenience of writing the final system of shallow water equations on a rotating sphere, we again use the time t instead of τ and introduce the parameters

$$r_0 = R_0^{-1}, \qquad f_0 = F^{-2}.$$

Then, the sought system acquires the form

$$\begin{gathered} D_0 v = w^2 \cot\,\theta + r_0 w \cos\theta + (1/4) r_0^2 \sin\theta\cos\theta - f_0 h_\theta, \\ D_0 w = -vw \cot\,\theta - r_0 v \cos\theta - f_0 (\sin\theta)^{-1} h_\varphi, \qquad D_0 h + (\sin\theta)^{-1} h (w_\varphi + (v\sin\theta)_\theta) = 0, \end{gathered} \tag{3.13}$$

where $D_0 = \partial_t + v\,\partial_\theta + (\sin\theta)^{-1} w\,\partial_\varphi$ is the total derivative on the sphere. Equations (3.13) are satisfied on the sphere surface in a gas layer whose thickness is small, as compared with the sphere radius.

Let us note some general properties of system (3.13).

1. Equations (3.13) coincide with the equations of gas dynamics for a polytropic gas in the case of isentropic motions of a special form, which occur on the surface of the rotating sphere; in this case, the radial component of velocity equals zero, and all sought functions are independent of the quantity r. Such motions of the gas may be called spherical motions. For $\rho = h$, the equation of state has the form $p = f_0\rho^2/2$. Hence, system (3.13) is hyperbolic and inherits all properties of gas-dynamic equations, in particular, the presence of acoustic characteristics.

2. System (3.13) is determined on a sphere with punctured points $\theta = 0$ and π in the poles. These singular points are points of intersection of the axis of revolution with the sphere surface. Thus, the equations describing the motion already have two singularities, which have the physical meaning, namely, the singularities of the flow in the poles.

4. Evaluation of the Boundaries of the Area of Applicability of the Model. For the Earth, we have $\Omega_0 = 7.3 \cdot 10^{-5}$ sec^{-1} and $a_0 = 6.4 \cdot 10^6$ m. Let us consider the motions with the characteristic durations from one day to several weeks, i.e., $T_0 \sim 10^6$ sec (1 day = 86,400 sec). For such motions, we have $H_0 \sim 10^3$ and $V_0 \sim 10$ m/sec. Then, we obtain $U_0 \sim 10^{-3}$ m/sec. Let us estimate the Rossby and Froude numbers:

$$R_0^{-1} = 2\Omega_0 a_0/V_0 \approx 90, \qquad F^{-2} = gH_0/V_0^2 \approx 10^2.$$

Hence, the parameters $r_0 = R_0^{-1}$ and $f_0 = F^{-2}$ in Eqs. (3.13) indeed have the same order, and rotation and gravity corresponding to these parameters exert comparable effects on medium motion. The parameter ε and the vertical velocity are negligibly small, as compared with the parameters r_0 and f_0.

5. Vorticity Equation. The vector velocity field tangential to the sphere $\boldsymbol{u} = (0, v, w)$, which satisfies system (3.13), has the vorticity $\boldsymbol{\omega} = \operatorname{rot} \boldsymbol{u} = (\omega, 0, 0)$ directed along the radius:

$$\omega = (\sin\theta)^{-1}((w\sin\theta)_\theta - v_\varphi) = w_\theta - (\sin\theta)^{-1} v_\varphi + w\cot\,\theta. \tag{5.1}$$

To derive the vorticity equations, we write the first two equations of system (3.13) in the form of the Gromeka–Lamb equations:

$$v_t - w\Omega_1 + G_\theta = 0, \qquad w_t + v\Omega_1 + (\sin\theta)^{-1} G_\varphi = 0. \tag{5.2}$$

Here, $\Omega_1 = \omega + r_0\cos\theta$ and $G = f_0 h + (v^2 + w^2)/2 - (1/8) r_0^2 \sin^2\theta$. The quantity $\Omega = h^{-1}\Omega_1$ is called the potential vorticity [3]. Then, the following lemma is valid.

Lemma 1. *The potential vorticity is conserved along the trajectories*

$$D_0\Omega = 0. \tag{5.3}$$

Proof. We multiply the second equation in (5.2) by $\sin\theta$, differentiate the first equation in (5.2) with respect to φ, differentiate the second equation in (5.2) with respect to θ, and subtract the first equation from the second one. Multiplying the resultant relation by $(\sin\theta)^{-1}$ and changing the sign, we obtain

$$(w_\theta - (\sin\theta)^{-1} v_\varphi + w\cot\,\theta)_t + v\Omega_{1\theta} + (\sin\theta)^{-1} w\Omega_{1\varphi} + (v_\theta + (\sin\theta)^{-1} w_\varphi + v\cot\,\theta)\Omega_1 = 0$$

or

$$D_0\Omega_1 + \Omega_1 \operatorname{div} \boldsymbol{u} = 0, \tag{5.4}$$

where $\operatorname{div} \boldsymbol{u} = v_\theta + (\sin\theta)^{-1} w_\varphi + v \cot\theta$. In system (3.13), the equation for h has the same form that Eq. (5.4):

$$D_0 h + h \operatorname{div} \boldsymbol{u} = 0. \tag{5.5}$$

Hence, multiplying Eq. (5.4) by h, multiplying Eq. (5.5) by Ω_1, and summing the resultant equations, we find

$$D_0(\Omega_1/h) = 0.$$

Lemma 1 is proved.

Lemma 1 extends the statement about the behavior of potential vorticity, corresponding to this lemma in the "plane" shallow water theory, to the spherical case.

6. Stationary Motions of Shallow Water. Let us consider stationary motions of shallow water on a rotating sphere, which are described by the equations

$$\begin{gathered} D_s v = w^2 \cot\theta + r_0 w \cos\theta + (1/4) r_0^2 \sin\theta\cos\theta - f_0 h_\theta, \\ D_s w = -vw \cot\theta - r_0 v \cos\theta - f_0 (\sin\theta)^{-1} h_\varphi, \\ D_s h + (\sin\theta)^{-1} h (w_\varphi + (v \sin\theta)_\theta) = 0, \end{gathered} \tag{6.1}$$

where $D_s = v\,\partial_\theta + (\sin\theta)^{-1} w\,\partial_\varphi$ is the derivative along the streamlines. Equations (6.1) include the Bernoulli integral and the integral of conservation of potential vorticity. The Bernoulli integral is obtained if the first equation of system (6.1) multiplied by v is summed with the second equation multiplied by w. Conservation of potential vorticity follows from Lemma 1.

To write these integrals, it is convenient to introduce the stream function $\psi = \psi(\theta, \varphi)$. As the third equation of system (6.1) is written in divergent form

$$(hv \sin\theta)_\theta + (hw)_\varphi = 0,$$

it can be satisfied by introducing the stream function ψ and assuming that

$$v = \psi_\varphi / (h \sin\theta), \qquad w = -\psi_\theta / h. \tag{6.2}$$

In this case, we have $D_s \psi = 0$. Let us use ∇_s and Δ_s to denote the gradient and Laplace operators on a unit sphere:

$$\nabla_s = \Big(\frac{\partial}{\partial\theta}, (\sin\theta)^{-1} \frac{\partial}{\partial\varphi}\Big), \qquad \Delta_s = (\sin\theta)^{-1} \frac{\partial}{\partial\theta}\Big(\sin\theta \frac{\partial}{\partial\theta}\Big) + (\sin\theta)^{-2} \frac{\partial^2}{\partial\varphi^2}.$$

By calculating the vorticity Ω_1 in terms of the stream function ψ and taking into account Eqs. (5.1) and (6.2), we obtain

$$\Omega_1 = \omega + r_0 \cos\theta = -(\sin\theta)^{-1} \Big((\psi_\theta \sin\theta / h)_\theta + (\sin\theta)^{-1} (\psi_\varphi / h)_\varphi\Big) + r_0 \cos\theta.$$

Let us denote $q = h^{-1}$. Then, the expression for Ω_1 acquires the form

$$\Omega_1 = -(\sin\theta)^{-1} [q((\psi_\theta \sin\theta)_\theta + (\sin\theta)^{-1} \psi_{\varphi\varphi}) + \psi_\theta q_\theta \sin\theta + (\sin\theta)^{-1} \psi_\varphi q_\varphi] + r_0 \cos\theta = -q\,\Delta_s \psi - \nabla_s \psi \cdot \nabla_s q + r_0 \cos\theta.$$

In the shallow water model, the equations of stationary motions

$$D_s((v^2 + w^2)/2 + f_0 h - (1/8) r_0^2 \sin^2\theta) = 0, \qquad D_s(\Omega_1 / h) = 0$$

integrated in terms of the stream function take the form

$$\begin{gathered} q^2 |\nabla_s \psi|^2 / 2 + f_0 / q - (1/8) r_0^2 \sin^2\theta = B_0(\psi), \\ q\,\Delta_s \psi + \nabla_s \psi \cdot \nabla_s q - r_0 \cos\theta = A_0(\psi), \end{gathered} \tag{6.3}$$

where A_0 and B_0 are arbitrary functions of their arguments. System (6.3) provides a complete description of stationary flows of shallow water on a rotating sphere and is equivalent to system (6.1) for such flows. The functions A_0 and B_0 characterize the values of energy and potential vorticity in terms of the stream function ψ.

The first equation in system (6.3) is an algebraic equation of the third power with respect to q. Let us write in the following form:

$$q^3 - a_0 q + b_0 = 0. \tag{6.4}$$

Here, $a_0 = |\nabla_s \psi|^{-2}(2B_0 + r_0^2 \sin^2\theta)$ and $b_0 = 2|\nabla_s \psi|^{-2} f_0 > 0$. We may also assume that $a_0 > 0$. Otherwise, Eq. (6.4) has no positive solutions with respect to q. The discriminant of Eq. (6.4) is

$$\mathcal{D} = 4a_0^3 - 27b_0^2. \tag{6.5}$$

Hence, Eq. (6.4) has three real roots for $\mathcal{D} > 0$. According to the Viette theory, two of these roots are positive, while one root is negative and has to be omitted because $q = 1/h > 0$. The case where Eq. (6.4) has one real root for $\mathcal{D} < 0$ is also impossible, because this root has to be negative, as is predicted by the Viette theorem.

If discriminant (6.5) is positive and Eq. (6.4) has three real roots, then two of them are positive and correspond to two different regimes of stationary motions. Thus, the following lemma is proved.

Lemma 2. *For stationary flows of shallow water on a rotating sphere, two regimes of motions corresponding to two different positive roots of Eq.* (6.4) *are possible.*

7. Equivalence Transformation of Stationary Equations. For two-dimensional equations of hydrodynamics, the stream function is determined with functional arbitrariness: its arbitrary function also defines the stream function. A similar (but slightly more complicated) situation occurs in the case of the stationary shallow water equations (6.3).

Theorem 1. *Let the set* (ψ, q, f_0, r_0) *define the solution of Eqs.* (6.3) *with specified functions* A_0 *and* B_0*,* $\chi = \chi(z)$*,* $\chi' > 0$ *is an arbitrary smooth, rigorously monotonic function, and* $Q_0 \neq 0$ *is a constant.*

The quantities ψ_1*,* q_1*,* f_{01}*, and* r_{01} *and the functions* A_{01} *and* B_{01} *are defined as follows:*

$$\psi = \chi(\psi_1), \quad q = \frac{Q_0}{\chi'(\psi_1)}\, q_1, \quad f_0 = \frac{Q_0^3}{\chi'(\psi_1)}\, f_{01}, \quad r_0 = Q_0 r_{01},$$
$$A_{01} = Q_0^{-1} A_0, \qquad B_{01} = Q_0^{-2} B_0 \tag{7.1}$$

(*the prime indicates the derivative with respect to the argument* ψ_1). *Then, the set* $(\psi_1, q_1, f_{01}, r_{01})$ *defines the solution of Eqs.* (6.3) *with the functions* A_{01} *and* B_{01} *in the right sides.*

Remark 1. Formulas (7.1) define an infinitely dimensional Lie group of equivalence of Eqs. (6.3), which depends only on one arbitrary function χ of one argument and one constant Q_0. The emergence of this group is explained by functional arbitrariness in choosing the stream function. Equations (7.1) show how other parameters of the problem are transformed thereby. It turns out that such a replacement involves transformation of the Froude number $F = f_0^{-2}$ with functional arbitrariness.

Particular cases of this transformation have a simple physical meaning. For $f_0 = r_0 = 0$, system (6.3) describes the inertial stationary shallow water flows on a spherical surface without rotation. In this case, Eqs. (7.1) define the transformation of system (6.3) with functional arbitrariness and can be used for solution multiplication: based on the known solution ψ_1, Eqs. (7.1) define a new solution.

For $f_0 \neq 0$, $r_0 \neq 0$, and a linear function $\xi(\psi_1) = \xi_0 \psi_1$ ($\xi_0 = \text{const}$), formulas (7.1) define the scale transformations (6.3). With an appropriate choice of the constants

$$Q_0 = r_0, \qquad \xi_0 = r_0^3/f_0,$$

the parameters r_0 and f_0 can be made equal to unity. In this case, the Rossby number r_0 and Froude number f_0 appear in the right side of Eqs. (6.3) as multipliers at the functions A_0 and B_0.

Remark 2. It is known that stationary equations of gas dynamics with distributed density admit an infinitely dimensional Lie group of equivalence, which is called the Munk–Prim transform [12]. This group transforms the Bernoulli function (right side of the Bernoulli integral), entropy, and the equation of state $\rho = a(s)b(p)$. By virtue of the above-noted gas-dynamic analogy, shallow water equations coincide with equations of isentropic gas dynamics with a polytropic equation of state with $\gamma = 2$ (for special solutions). Shallow water equations, however, do not admit the Munk–Prim transform directly, because these transformations change the equation of state, which has a fixed form $p = \rho^2/2$ ($\rho = h$) for shallow water equations. Nevertheless, the stationary equations (6.3) admit the infinitely dimensional Lie group (7.1), which transforms the parameter f_0 in this equation of state.

Proof of Theorem 1. The theorem is proved by means of direct check. Let $\psi = \chi(\psi_1)$ and $q = Q(\psi_1) q_1$, where χ and Q are certain functions of ψ_1. Then, we obtain the following formulas after omission of the index s at differential operators:

$$\nabla\psi = \chi' \nabla\psi_1, \quad \Delta\psi = \chi' \Delta\psi_1 + \chi'' |\nabla\psi_1|^2, \quad \nabla q = Q \Delta q_1 + Q' q_1 \nabla\psi_1. \tag{7.2}$$

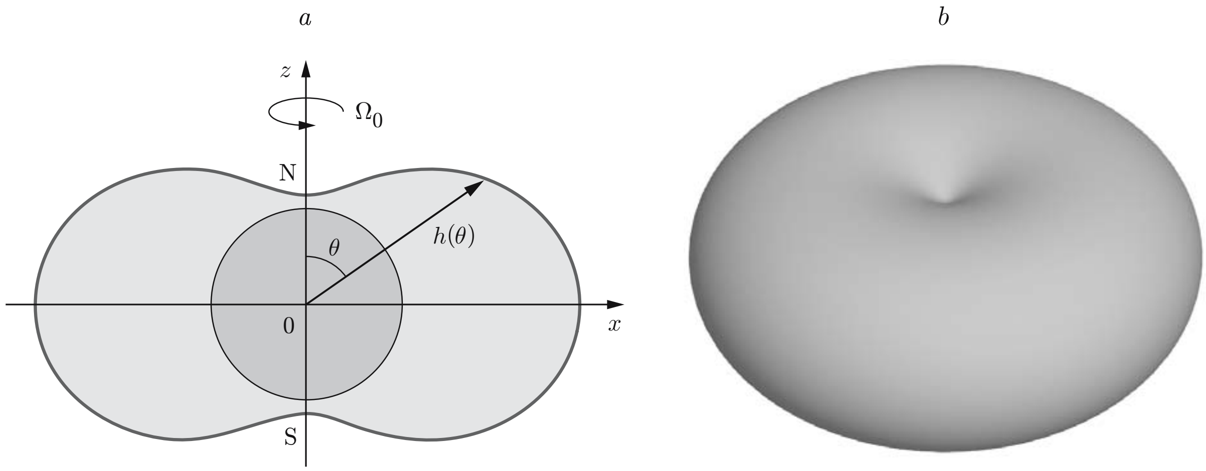

Fig. 2. Equilibrium surface: (a) diametral cross section; (b) three-dimensional view.

Substituting Eq. (7.2) into system (6.3) as

$$\begin{gathered}\frac{Q^2q_1^2}{2}\,\chi'^2|\nabla\psi_1|^2+\frac{f_0}{Qq_1}-\frac{1}{8}\,r_0^2\sin^2\theta=B_0,\\ Qq_1\chi'\,\Delta\psi_1+(Q\chi''+Q'\chi')q_1|\nabla\psi_1|^2+Q\chi'\nabla q_1\cdot\nabla\psi_1-r_0\cos\theta=A_0,\end{gathered}\tag{7.3}$$

expressing $|\nabla\psi_1|^2$ from the first equation in (7.3), and substituting it into the second equation, we obtain

$$Qq_1\chi'\,\Delta\psi_1+Q\chi'\,\nabla q_1\cdot\nabla\psi_1-r_0\cos\theta+\frac{2(Q\chi''+Q'\chi')}{q_1Q^2\chi'^2}\Big(B_0+\frac{1}{8}\,r_0^2\sin^2\theta-\frac{f_0}{Qq_1}\Big)=A_0.\tag{7.4}$$

It follows from Eq. (7.4) that the second equation of (6.3) with $Q\chi'=Q_0=\text{const}$ transforms to an equation of the same form. Dividing the first equation in (7.3) by $Q^2q_1^2/2$ and equation (7.4) by Qq_1, with $Q=Q_0(\chi')^{-1}$, we obtain formulas (7.1). Theorem 1 is proved.

8. Simple Solutions. An important feature of the shallow water model derived is the presence of an equilibrium state with zero relative components of velocity ($u=w=0$) and with the following distribution of depth:

$$h=\alpha_0^2(k_0^2+\sin^2\theta).\tag{8.1}$$

Here $\alpha_0^2=r_0^2/(8f_0)$ and $k_0^2=8f_0h_0/r_0^2$ ($h_0>0$) are constants. The velocity of propagation of acoustic disturbances on solution (8.1) is $c=(r_0/2\sqrt{2}\,)(k_0^2+\sin^2\theta)^{1/2}$. For $\theta\in(0,\pi)$, the equation

$$r=\alpha_0^2(k_0^2+\sin^2\theta)\tag{8.2}$$

in the space $\mathbb{R}^3(\boldsymbol{x})$ defines the surface of revolution, characterizing the equilibrium profile of depth, which differs from the spherical profile. Figure 2 shows the surface described by Eq. (8.2).

Let us consider stationary motions of shallow water with both components of velocity v and w and depth h depending only on the latitude θ. In this case, system (6.1) reduces to a system of ordinary differential equations

$$\begin{gathered}vv'=w^2\cot\theta+r_0w\cos\theta+(r_0^2/4)\sin\theta\cos\theta-f_0h',\\ vw'=-vw\cot\theta-r_0v\cos\theta,\qquad vh'\sin\theta+h(v\sin\theta)'=0,\end{gathered}\tag{8.3}$$

where the prime denotes the derivative with respect to θ. System (8.3) is integrated in its finite form.

There are two types of solutions: $v\equiv0$ in solutions of the first type and $v\neq0$ in solutions of the second type. Let us consider solutions of the first type. In this case, the second and third equations of system (8.3) are identically satisfied, and this system reduces to one equation

$$w^2+r_0w\sin\theta+(r_0^2/4)\sin^2\theta-f_0h'\tan\theta=0\tag{8.4}$$

relating the depth h to the circumferential component of velocity w. It follows from Eq. (8.4) that

$$w_{1,2} = -(r_0 \sin\theta + 2\varepsilon(f_0 h' \tan\,\theta)^{1/2})/2 \qquad (\varepsilon = \pm 1). \tag{8.5}$$

Solutions (8.5) are determined at $h' \tan\theta > 0$; hence, as $\sin\theta > 0$ for $\theta \in (0, \pi)$, we obtain $h' > 0$ [$\theta \in (0, \pi/2)$] in the north hemisphere and $h' < 0$ [$\theta \in (\pi/2, \pi)$] in the south hemisphere. This solution describes the flow along the parallels with an arbitrary depth profile, which monotonically increases in the north hemisphere and decreases in the south hemisphere with increasing latitude. Solutions of the form (8.5) model flows, such as jet motions in the atmosphere, which propagate predominantly along the parallels. In the Earth's atmosphere, such flows arise on cell boundaries. Thus, the polar front of jet flows is located between the polar cells and the Ferrel cells, whereas subtropical jet flows separate the Ferrel cells and the Hadley cells [13]. In these convective cells existing in the Earth's atmosphere, the air masses from the equatorial region move to the north in the north hemisphere and to the south in the south hemisphere, deviating from meridians due to the Coriolis force. When a certain critical pressure is reached, a flow moving in the opposite direction (toward the equator) arises above the initial flow.

Let us consider solutions of type (8.5), which are critical everywhere. In other words, the equality $w^2 = f_0 h$ is satisfied in the entire domain of solution definition, which implies that the velocity of propagation of acoustic disturbances in the layer equals the velocity of particle motion along the streamlines. Then, Eq. (8.5) yields the differential equation for the function h

$$h' \tan\,\theta = 4 f_0 \Big(h - \frac{r_0}{4 f_0}\,\sin\theta\Big)^2, \tag{8.6}$$

which is the Riccati equation. Introducing a new function z instead of h by the formula

$$h = \frac{r_0 \sin\theta}{4 f_0}\,\Big(1 - \frac{z'}{z r_0 \cos\theta}\Big), \tag{8.7}$$

we transform Eq. (8.6) to the second-order linear equation

$$z'' + \frac{1}{\sin\theta\cos\theta}\,z' - \frac{r_0 \cos^2\theta}{\sin\theta}\,z = 0. \tag{8.8}$$

Applying the substitution $y = \sin\theta$ in Eq. (8.8), we obtain a modified Bessel equation with $\nu = 0$ [14]:

$$y z_{yy} + z z_y - r_0 z = 0. \tag{8.9}$$

Solution (8.9) has the form

$$z = C_1 I_0(\xi) + C_2 K_0(\xi), \tag{8.10}$$

where I_0 and K_0 are the zero-order modified Bessel functions of the argument $\xi = 2\sqrt{r_0 y}$; C_1 and C_2 are arbitrary constants.

Substituting the value of z calculated by Eq. (8.10) into Eq. (8.7), we obtain a formula of the form (8.5) for the depth profile in the critical flow:

$$h = \frac{r_0 \sin\theta}{4 f_0}\Big(1 - \frac{\cos\theta}{r_0}\,\frac{\sqrt{r_0}\,(C_1 I_1(\xi) - C_2 K_1(\xi))}{\sqrt{\sin\theta}\,(I_0(\xi) C_1 + K_0(\xi) C_2)}\Big). \tag{8.11}$$

Here $\xi = 2\sqrt{r_0 \sin\theta}$; I_1 and K_1 are the first-order modified Bessel functions of the first and second kind. Note that the replacement $t = 1/y$ transforms Eq. (8.9) to the Kelvin equation and the corresponding modified Bessel functions to the Kelvin functions. Solution (8.11) has to satisfy the above-formulated condition of monotonicity $h' \cos\theta > 0$. The velocity w is calculated by Eq. (8.5).

Conclusions. Equations of the shallow water model on a rotating attracting sphere are derived in the paper. This model describes the motions of a continuous medium (air in the planetary atmosphere and liquid in the world ocean) of global (planetary) scales, the thickness of the layer where the motions occur being small as compared with the scales of motions on the planet surface. Because of the small thickness of the layer, the velocity of the medium can be considered as its averaged value over the depth of the layer.

The system of equations derived coincides with equations of isentropic polytropic gas dynamics in the case of gas motions on the surface of a rotating sphere. The system is hyperbolic and determined on a compact manifold with singularities in the sphere poles.

Conservation of potential vorticity along the trajectories is proved. For stationary flows, an equivalent system of two equations in terms of the layer thickness and stream function is obtained; the existence of two regimes of stationary motions of shallow water with different depth profiles is proved. An infinitely dimensional Lie group of equivalence is found, which transforms the stream function, the depth, and the Froude number of the flow. The simplest stationary solutions of the model, which correspond to the equilibrium state and differ from spherically symmetric and zonal flows along the parallels, are described.

The model obtained seems to be promising for studying large-scale motions. An important feature of this model is the possibility of using it to study results obtained in gas dynamics. A natural first step in this direction is the search for simple exact solutions.

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